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**COOPERATIVE GAMES ON ANTIMATROIDS**

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# Cooperative games on antimatroids

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## Abstract

The aim of this paper is to introduce cooperative games with a feasible coalition system which is called *antimatroid*. These combinatorial structures generalize the *permission structures*, which have nice economical applications. With this goal, we first characterize the approaches from a permission structure with special classes of antimatroids. Next, we use the concept of *interior operator* in an antimatroid and we define the restricted game taking into account the limited possibilities of cooperation determined by the antimatroid. These games extend the restricted games obtained by permission structures. Finally, we provide a computational method to obtain the *Shapley* and *Banzhaf values* of the players in the restricted game, by using the worths of the original game.

*AMS 1991 subject classification:* 90D12

*Key words:* Permission structures, antimatroids, Shapley and Banzhaf values

## 1 Introduction

A *TU-game* or *transferable utility game* describes a situation in which a finite set of players  $N$  can generate certain payoffs by cooperation. We will denote by  $\Gamma^N$  the set of all TU-games on  $N$ . In a TU-game the players are assumed to be socially identical in the sense that every player can cooperate with every other player. However, in practice there exist social asymmetries among the players. For this reason, the game theoretic analysis of decision processes in which one imposes asymmetric constraints on the behavior of the players has been and continue being an important subject to make a study. So, important consequences have been obtained of adopting this type of restrictions on economic behavior. Some models, in which have been analyzed social asymmetries among players in a TU-game, are described in e.g. Myerson (1977), Owen (1986) and Borm, Owen and Tijs (1992). In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph*.

Another type of asymmetry among the players in a TU-game is introduced in Gilles, Owen and van den Brink (1992), Gilles and Owen (1999) and van den Brink and Gilles (1996). In these models, the possibilities of coalition formation are determined by the positions of the players in the so-called *permission structure*. Other related models can be found in Faigle and Kern (1992) and Derks and Peters (1993).

In the present paper, we use the restricted cooperation model derived from an *antimatroid*. Section 2 introduces permission structures and antimatroids and we

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show that given a permission structure, the approaches from it are an antimatroid but not every antimatroid is an approach from a permission structure. Moreover, we identify the approaches from an acyclic permission structure with antimatroids satisfying specific properties. This study gives rise to a new class of antimatroids obtained through permission structures. Two new concepts in these structures are essential, on one hand the *path property* and on the other hand the *feasible hull* which are both based in the path characterization of antimatroids. Section 3 introduces the restricted games on antimatroids which generalize the ones studied on permission structures. Next, we study in detail some properties of these games, above all those properties that are transmitted from the original game to the restricted game, and we give a description of the dividends in such games. Using the structural properties from the antimatroid we will be able to express the dividends in terms of the original game. This result will be essential in the last section to provide some formulas to compute the Shapley and Banzhaf values for restricted games on antimatroids. In these formulas these values are computed by means of the original game without having to calculate the restricted game and taking into account only the coalitions in the antimatroid.

## 2 Permission structures and antimatroids

Permission structures were defined by Gilles *et al.* (1992). They assume that players who participate in a TU-game are part of a hierarchical organization in which there are players that need permission from certain other players before they are allowed to cooperate.

For a finite set of players  $N = \{1, 2, \dots, n\}$  such a hierarchical organization is given by a mapping  $S : N \rightarrow 2^N$  which is called a *permission structure* on  $N$ . The players in  $S(i)$  are called the *successors* of player  $i$  in the permission structure  $S$  and the players in  $S^{-1}(i) = \{j \in N : i \in S(j)\}$  are named the *predecessors* of  $i$  in  $S$ . A *chain of players* is an ordered list  $(h_1, \dots, h_t)$  where  $h_{k+1} \in S(h_k)$ , for all  $k = 1, \dots, t-1$ . The *transitive closure* of a player  $i$  in  $S$  is the set

$$\hat{S}(i) = \{j : \text{there exists a chain } (h_1, \dots, h_t) \text{ with } h_1 = i, h_t = j\},$$

whose players are called the *subordinates* of player  $i$  in  $S$ . The players in the set  $\hat{S}^{-1}(i) = \{j \in N : i \in \hat{S}(j)\}$  are named the *superiors* of  $i$  in  $S$ . A permission structure  $S$  on  $N$  is *acyclic* if  $i \notin \hat{S}(i)$ , for all  $i \in N$ .

In Gilles *et al.* (1992) the conjunctive approach to games with a permission structure is defined. In this approach it is assumed that each player needs permission from all his predecessors before it is allowed to cooperate with other players. Alternatively, in the disjunctive approach as discussed in van den Brink (1997) and Gilles and Owen (1999) it is assumed that each player needs permission from at least one of his predecessors before he is allowed to cooperate with others players. Thus, the feasible coalitions in the conjunctive and disjunctive approaches respectively are given by the sets

$$\begin{aligned} \Phi_S^c &= \{E \subseteq N : \text{for every } i \in E \text{ it holds that } S^{-1}(i) \subseteq E\}, \\ \Phi_S^d &= \{E \subseteq N : \text{for every } i \in E, S^{-1}(i) \neq \emptyset \text{ it holds that } S^{-1}(i) \cap E \neq \emptyset\}. \end{aligned}$$

We will show that the feasible coalitions in the conjunctive and disjunctive approaches of permission structures are identified with special classes of set systems called antimatroids. *Antimatroids* were introduced by Dilworth (1940) as particular examples of semimodular lattices. Several authors have obtained the same concept by abstracting various combinatorial situations. A systematic study of these structures was started by Edelman and Jamison (1985) emphasizing the combinatorial abstraction of convexity. The latter was then shown by Edelman (1980) to be a crucial property of closures induced by what he called *convex geometries*, a dual concept

of antimatroids (see Bilbao, 2000). Jiménez-Losada (1998) introduced antimatroids in games, defining games on the coalitions of the set system given by the antimatroid.

**Definition 1** An antimatroid  $\mathcal{A}$  on  $N$  is a family of subsets of  $2^N$ , satisfying

**A1.**  $\emptyset \in \mathcal{A}$ .

**A2.** (Accessibility) If  $E \in \mathcal{A}$ ,  $E \neq \emptyset$ , then there exists  $i \in E$  such that  $E \setminus \{i\} \in \mathcal{A}$ .

**A3.** (Closed under union) If  $E, F \in \mathcal{A}$  then  $E \cup F \in \mathcal{A}$ .

Now, we need to introduce some well-known concepts about antimatroids, which can be found in Korte, Lovász and Schrader (1991).

The definition of antimatroid implies the following *augmentation property*, i.e., if  $E, F \in \mathcal{A}$  with  $|E| > |F|$  then there exists  $i \in E \setminus F$  such that  $F \cup \{i\} \in \mathcal{A}$ . From now on we assume that the antimatroid  $\mathcal{A}$  is *normal*, i.e.,

**A4.** For every  $i \in N$  there exists an  $E \in \mathcal{A}$  such that  $i \in E$ .

In particular, this last property implies that  $N \in \mathcal{A}$  and therefore this means that the whole group of players decide to cooperate.

Let  $\mathcal{A}$  be an antimatroid on  $N$ . This set family allows to define the *interior operator*  $\text{int} : 2^N \rightarrow \mathcal{A}$ , given by  $\text{int}(E) = \bigcup_{\{F \subseteq E, F \in \mathcal{A}\}} F \in \mathcal{A}$ , for all  $E \subseteq N$ . This operator is the dual one of the closure operator in a convex geometry and satisfies the following properties which characterize it:

**I1.**  $\text{int}(\emptyset) = \emptyset$ ,

**I2.**  $\text{int}(E) \subseteq E$ ,

**I3.** if  $E \subseteq F$  then  $\text{int}(E) \subseteq \text{int}(F)$ ,

**I4.**  $\text{int}(\text{int}(E)) = \text{int}(E)$ ,

**I5.** if  $i, j \in \text{int}(E)$  and  $j \in \text{int}(E \setminus \{i\})$  then  $i \notin \text{int}(E \setminus \{j\})$ .

Let  $\mathcal{A}$  be an antimatroid on  $N$ . A *feasible continuation* or *augmentation point* (Jiménez-Losada, 1998) of  $E \in \mathcal{A}$  is a player  $i \in N \setminus E$  such that  $E \cup \{i\} \in \mathcal{A}$ , i.e., those players that can be joined to a feasible coalition keeping feasibility. This elements are denoted as  $\text{au}(E)$ . In a dual way, a player that can leave a feasible coalition  $E$  keeping feasibility is called *endpoint* or *extreme point* (Edelman and Jamison, 1985). The endpoints of  $E$  are denoted by  $\text{ex}(E)$ . By the condition A2 the sets  $\text{ex}(E) \neq \emptyset$ , for all  $E \in \mathcal{A}$ ,  $E \neq \emptyset$ . In the same way, using antimatroid definition and A4, the sets  $\text{au}(E) \neq \emptyset$ , for all  $E \in \mathcal{A}$ ,  $E \neq N$ .

A set  $E$  in  $\mathcal{A}$  is a *path* if it has a single endpoint. So, the set  $E$  in  $\mathcal{A}$  is called a *i-path* if it has  $i$  as unique endpoint. A *i-path* can be considered as a rooted set denoted by  $(E, i)$ , i.e.,  $E$  is a minimal set in  $\mathcal{A}$  containing  $i$ . In particular, a set is in  $\mathcal{A}$  or is feasible in  $\mathcal{A}$  if and only if it is a union of paths. This leads to a characterization of antimatroids in terms of paths (see Korte *et al.*, 1991).

As a generalization of a rooted path in a player, we can give the following concept about coalitions.

**Definition 2** Let  $\mathcal{A}$  be an antimatroid on  $N$  and  $E \subseteq N$ . A set  $F \subseteq N$  is a *feasible hull* of  $E$  if  $F \in \mathcal{A}$ ,  $E \subseteq F$  and there exists no  $H \in \mathcal{A}$  with  $E \subseteq H \subset F$ , i.e., it is a feasible minimal set in  $\mathcal{A}$  and contains the set  $E$ .

We denote by  $A(E)$  the family of feasible hulls of  $E \subseteq N$ . Notice that  $A(E) \neq \emptyset$ . We show in the following result that this definition generalizes the path definition.

**Proposition 1** *Let  $\mathcal{A}$  be an antimatroid on  $N$ . The feasible hulls of  $\{i\}$ , with  $i \in N$  are the family of the  $i$ -paths of  $\mathcal{A}$ .*

**Proof.** If  $E$  is a feasible hull of  $\{i\}$ , then  $i \in E$  and there exists no  $F \in \mathcal{A}$  with  $i \in F \subset E$ . We have to prove that  $E$  is a  $i$ -path, by condition A2 there exists some endpoint of  $E$ , suppose that the possible endpoints  $j$  of  $E$  are different from  $i$ . In this case, it holds that  $i \in E \setminus \{j\} \subset E$  and as  $j$  is an endpoint  $E \setminus \{j\} \in \mathcal{A}$ , what led us to contradiction. So, the unique endpoint of  $E$  is  $i$ .

If  $E$  is a  $i$ -path and it is not a feasible hull of  $\{i\}$  then there exists  $F \in \mathcal{A}$  with  $i \in F$  and  $F \subset E$ . Applying the augmentation property, there exists  $j \in E \setminus \{i\}$  such that  $E \setminus \{j\} \in \mathcal{A}$ . This contradicts the fact that  $E$  is a  $i$ -path.  $\square$

Now we consider the relations between acyclic permission structures and antimatroids.

**Proposition 2** *If the feasible coalition system  $\mathcal{A}$  is derived from an disjunctive or conjunctive approach of an acyclic permission structure then  $\mathcal{A}$  is an antimatroid.*

**Proof.** Gilles *et al.* (1992), Gilles and Owen (1999) showed that the feasible coalitions system  $\mathcal{A}$  derived from the conjunctive or disjunctive approach contains the empty set and that it is closed under union. Therefore, it suffices to prove that the system  $\mathcal{A}$  is accessible. Let  $S$  be an acyclic permission structure on  $N$ . As  $N$  is a finite set, given a feasible coalition  $E$ ,  $E \neq \emptyset$ , there exists a player  $i \in E$  such that  $S(i) \cap E = \emptyset$  by acyclicity of  $S$ . Then, the coalition  $E \setminus \{i\}$  is also feasible, and hence we obtain the condition A2.  $\square$

The reverse of Proposition 2 is not true (see Example 1). This result will later be useful to generalize the restricted games on permission structures and introduce in the next section the restricted games on antimatroids. Next, some specific antimatroids are defined to characterize the feasible coalition systems derived from the conjunctive approach of an acyclic permission structure.

The so-called *poset antimatroids* are a particular case of antimatroid, which are formed by the ideals of a *poset* (partially ordered set)  $\mathcal{P}$  or, in an equivalent way, by the filters. Given  $N$  consider a poset  $\mathcal{P} = (N, \preceq)$  defined on  $N$ , a player  $i$  covers another player  $j$  in  $\mathcal{P}$ , if  $j \prec i$  and there is not another player  $h$  with  $j \prec h \prec i$ . The set  $I \subseteq 2^N$  is an *ideal* of  $\mathcal{P}$ , if for all  $i \in I$  and  $j \prec i$  implies that  $j \in I$ , and  $F$  is a *filter* if for all  $i \in F$  and  $j \succ i$  then  $j \in F$ . The poset antimatroids can be characterized as the unique antimatroids which are closed under intersection, i.e., if  $E, F \in \mathcal{A}$  then  $E \cap F \in \mathcal{A}$ .

**Lemma 1** *Let  $\mathcal{A}$  be an antimatroid.  $\mathcal{A}$  is a poset antimatroid if and only if every  $i \in N$  has a unique  $i$ -path in  $\mathcal{A}$ .*

**Proof.** Suppose that  $\mathcal{A}$  is a poset antimatroid. Let  $E, F$ ,  $E \neq F$ , be two distinct  $i$ -paths for  $i \in N$ . Then  $E \cap F \in \mathcal{A}$  with  $i \in E \cap F$ . Assume without loss of generality that  $E \setminus F \neq \emptyset$ . By the augmentation property there exists a  $j \in E \setminus (E \cap F) = E \setminus F \subseteq E \setminus \{i\}$  such that  $E \setminus \{j\} \in \mathcal{A}$ . This is in contradiction with  $E$  being a  $i$ -path.

Suppose that every  $i \in N$  has a unique  $i$ -path in  $\mathcal{A}$ . Take  $E, F \in \mathcal{A}$ . If  $E \cap F = \emptyset$  then  $E \cap F \in \mathcal{A}$  by A1. If  $E \cap F \neq \emptyset$  then by Proposition 1, for every  $i \in E \cap F$  there exists a  $i$ -path  $H_1^i \subseteq E$  and there exists a  $i$ -path  $H_2^i \subseteq F$ . By assumption  $H_1^i = H_2^i = H^i$  is the unique  $i$ -path in  $\mathcal{A}$ . So,  $H^i \in \mathcal{A}$  and  $H^i \subseteq E \cap F$ , for all  $i \in E \cap F$ . Therefore,  $E \cap F = \bigcup_{i \in E \cap F} H^i$  and by A3  $E \cap F \in \mathcal{A}$ . Thus,  $\mathcal{A}$  is a poset antimatroid.  $\square$

In the following result we identify the feasible coalition system derived from the conjunctive approach on an acyclic permission structure with a poset antimatroid.

**Theorem 1**  $\mathcal{A}$  is a poset antimatroid if and only if there is an acyclic permission structure  $S$  such that  $\mathcal{A} = \Phi_S^c$ .

**Proof.** Let  $\mathcal{A}$  be a poset antimatroid. For  $i \in N$ , denote the unique  $i$ -path in  $\mathcal{A}$  by  $P_i^{\mathcal{A}}$ . Now, define permission structure  $S : N \rightarrow 2^N$  by  $S^{-1}(i) = P_i^{\mathcal{A}} \setminus \{i\}$ , for all  $i \in N$ . We have to prove that  $S$  is acyclic. Suppose  $i \in \hat{S}(i)$  then there exists  $(h_1, \dots, h_t)$  with  $h_1 = i$ ,  $h_t = i$  and such that  $h_{k+1} \in S(h_k)$ , for all  $k = 1, \dots, t-1$ , i.e.,  $h_k \in P_{h_{k+1}}^{\mathcal{A}}$ . Moreover,  $P_k^{\mathcal{A}} \subseteq P_{h_{k+1}}^{\mathcal{A}}$ , for all  $k$ . Since  $P_{h_{k+1}}^{\mathcal{A}} \in \mathcal{A}$  by Proposition 1 we have that the unique  $h_k$ -path is contained in  $P_{h_{k+1}}^{\mathcal{A}}$ . But  $P_k^{\mathcal{A}} = P_{h_{k+1}}^{\mathcal{A}}$  is impossible and hence, we get  $P_i^{\mathcal{A}} \subsetneq P_i^{\mathcal{A}}$ . So,  $S$  is acyclic. We are left to show that  $\mathcal{A} = \Phi_S^c$ .

Let  $E \in \Phi_S^c$  then  $S^{-1}(i) \subseteq E$ , for all  $i \in E$ , i.e.,  $P_i^{\mathcal{A}} \subseteq E$ , for all  $i \in E$ , therefore  $\bigcup_{i \in E} P_i^{\mathcal{A}} \subseteq E$ . Since  $i \in P_i^{\mathcal{A}}$ , for all  $i \in N$  we also have  $E \subseteq \bigcup_{i \in E} P_i^{\mathcal{A}}$ . Thus,  $E = \bigcup_{i \in E} P_i^{\mathcal{A}} \in \mathcal{A}$  since a union of paths is feasible by A3.

Clearly, if  $E \in \mathcal{A}$  with  $i \in E$  then  $S^{-1}(i) = P_i^{\mathcal{A}} \setminus \{i\} \subseteq E$ , and thus  $E \in \Phi_S^c$ .

By Proposition 2, the feasible coalition system from a conjunctive approach of an acyclic permission structure is an antimatroid. Gilles *et al.* (1992) showed that the feasible coalition system from the conjunctive approach is closed under intersection, therefore this approach is a poset antimatroid.  $\square$

Now we are interested in characterizing the antimatroids that can be the set of disjunctive feasible coalitions of some acyclic permission structure. With this goal we introduce the following definition.

**Definition 3** An antimatroid  $\mathcal{A}$  satisfies the path property if it is verified

**P1.** If  $E$  is a  $i$ -path then  $E \setminus \{i\}$  is a path.

**P2.** Let  $E$  be a  $i$ -path such that  $E \setminus \{i\}$  is a  $j$ -path. If  $F$  is another  $j$ -path in  $\mathcal{A}$  then  $F \cup \{i\} \in \mathcal{A}$  and  $i \notin F$ .

A graphic interpretation of the path property can be given by the order defined for the feasible coalitions of an antimatroid. Let  $\mathcal{A}$  an antimatroid and  $E \in \mathcal{A}$ , we have  $i \leq_E j$  when  $j \notin \text{int}(E \setminus \{i\})$ , for all  $i, j \in E$ . An antimatroid verifies the path property if and only if the order defined for each path is a total order and moreover if two any players have a established order then they have the same order for every feasible coalition. Next, we identify those antimatroids that satisfy the path property in relation to the feasible coalition systems derived from acyclic permission structures.

**Theorem 2**  $\mathcal{A}$  is an antimatroid satisfying the path property if and only if there is an acyclic permission structure  $S$  such that  $\mathcal{A} = \Phi_S^d$ .

**Proof.** Let  $\mathcal{A}$  be an antimatroid satisfying the path property. Define  $S : N \rightarrow 2^N$  by  $S^{-1}(i) = \{j \in N : \text{there exists an } E \in \mathcal{A}(\{i\}) \text{ such that } E \setminus \{i\} \in \mathcal{A}(\{j\})\}$ . Since  $\mathcal{A}$  satisfies the path property,  $S$  is acyclic. Now we prove that  $\mathcal{A} = \Phi_S^d$ .

If  $E \in \mathcal{A}$  we have that for every  $i \in E$  there exists a  $i$ -path  $F \subseteq E$  in  $\mathcal{A}$ . Thus, for every  $i \in E$ ,  $\{i\} \in \mathcal{A}$  or there exists a  $i$ -path  $F \subseteq E$  and a  $j \in F \setminus \{i\}$  such that  $F$  is a  $j$ -path in  $\mathcal{A}$ . Therefore, for every  $i \in E$ , as  $\{i\}$  is the unique  $i$ -path,  $S^{-1}(i) = \emptyset$  or  $S^{-1}(i) \cap E \neq \emptyset$  and hence  $E \in \Phi_S^d$ .

Suppose that  $E \in \Phi_S^d$ . We prove that  $E \in \mathcal{A}$  by induction on  $|E|$ . If  $|E| = 1$ , i.e.,  $E = \{i\}$ , then  $S^{-1}(i) = \emptyset$ . Thus, the unique path is  $E = \{i\} \in \mathcal{A}$ . Proceeding by induction, assume that  $E' \in \mathcal{A}$  if  $|E'| < |E|$ . Acyclicity of  $S$  implies that there exists a  $i \in E$  with  $S(i) \cap E = \emptyset$ . Then  $E \setminus \{i\} \in \Phi_S^d$ , and by the induction hypothesis  $E \setminus \{i\} \in \mathcal{A}$ .

Take a  $j \in S^{-1}(i) \cap E \neq \emptyset$  (otherwise,  $S^{-1}(i) = \emptyset$ , for all  $i \in E$ , but then it is obvious that  $E \in \mathcal{A}$ ). Since  $E \setminus \{i\} \in \mathcal{A}$  there exists a  $j$ -path  $F \subseteq E \setminus \{i\}$ . By  $\mathcal{A}$

satisfying the path property it holds that  $F \cup \{i\} \in \mathcal{A}$ . But then A3 implies that  $F \cup \{i\} \cup E \setminus \{i\} = E \in \mathcal{A}$ .

Let  $S$  be an acyclic permission structure. By Proposition 2,  $\mathcal{A} = \Phi_S^d$  is an antimatroid. We have to prove that  $\mathcal{A}$  satisfies the path property.

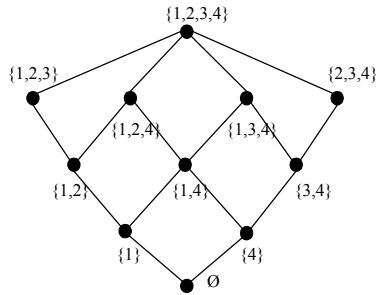
We first prove that if  $E$  is a  $i$ -path in  $\Phi_S^d$  then  $S(i) \cap E = \emptyset$  and  $S^{-1}(j) = \emptyset$  or  $|S^{-1}(j) \cap E| = 1$ , for all  $j \in E$ . Indeed, let  $E$  be a  $i$ -path in  $\Phi_S^d$ .

Suppose that  $S(i) \cap E \neq \emptyset$ . By acyclicity of  $S$  there exists a  $j \in \widehat{S}(i) \cap E$  such that  $S(j) \cap E = \emptyset$ . But then  $E \setminus \{j\} \in \Phi_S^d$ . Since  $j \neq i$  this is in contradiction with  $E \setminus \{i\}$  being a  $i$ -path in  $\Phi_S^d$ .

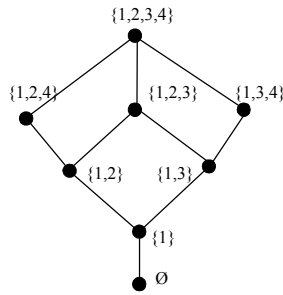
Suppose there exists a  $j \in E$  with  $|S^{-1}(j) \cap E| \geq 2$ . We distinguish the following two cases:

1. Suppose there exists an  $h \in E \setminus \{i\}$  with  $S(h) \cap E = \emptyset$ . Then  $E \setminus \{h\} \in \Phi_S^d$ , which is in contradiction with  $E$  being a  $i$ -path.
2. Suppose that  $S(h) \cap E \neq \emptyset$ , for all  $h \in E \setminus \{i\}$ . Then, there exists a  $g \in E$  with  $|S^{-1}(g) \cap E| \geq 2$  and  $|S^{-1}(h) \cap E| = 1$  for all  $h \in \widehat{S}(g) \cap E$ . Since  $S(i) \cap E = \emptyset$ ,  $i \notin S^{-1}(g) \cap E$ . By acyclicity of  $S$  it holds that  $S(k) \cap E = \{g\}$ , for all  $k \in S^{-1}(g) \cap E$ . But then  $E \setminus \{k\} \in \Phi_S^d$ , for all  $k \in S^{-1}(g) \cap E$ . This gives a contradiction with  $E$  being a  $i$ -path.

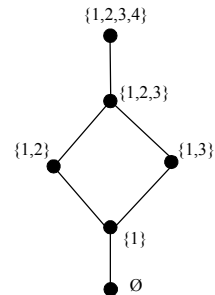
Therefore, we have that if  $E$  is a  $i$ -path in  $\Phi_S^d$  then  $E \setminus \{i\}$  is a path in  $\Phi_S^d$ . To be more specific, assume that  $E \setminus \{i\}$  is a  $j$ -path in  $\Phi_S^d$  with  $\{j\} = S^{-1}(i) \cap E$ . But then,  $F$  being a  $j$ -path in  $\Phi_S^d$  implies  $S^{-1}(h) = \emptyset$  or  $|S^{-1}(h) \cap F| = 1$ , for all  $h \in F$  and  $S(j) \cap F = \emptyset$ . Hence,  $S^{-1}(h) = \emptyset$  or  $|S^{-1}(h) \cap F| = 1$ , for all  $h \in F$  and  $S^{-1}(i) \cap F = \{j\} \neq \emptyset$ . Thus,  $F \cup \{i\} \in \mathcal{A}$ . On the other hand, as  $i \in S(j)$  and  $S(j) \cap F = \emptyset$  we have that  $i \notin F$ . So,  $\Phi_S^d$  satisfies the path property.  $\square$



Antimatroid  
Figure 1



Antimatroid with  
the path property  
Figure 2



Poset Antimatroid  
Figure 3

These examples show that not every antimatroid satisfies the path property.

**Example 1** Let  $N = \{1, 2, 3, 4\}$  and the following family of subsets of Figure 1,

$$\mathcal{A} = \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{1, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\},$$



then it is easy to see that  $\mathcal{A}$  is an antimatroid and it does not satisfy the path property, since  $E = \{2, 3, 4\}$  is a 2-path such that  $E \setminus \{2\} = \{3, 4\}$  is a 3-path. However,  $F = \{1, 2, 3\}$  is another 3-path and  $2 \in F$ .

**Example 2** Consider the antimatroid  $\mathcal{A}$  on  $N = \{1, 2, 3, 4\}$  of Figure 2,

$$\mathcal{A} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$

It can be checked that  $\mathcal{A}$  satisfies the path property.

**Example 3** Consider the poset antimatroid  $\mathcal{B}$  on  $N = \{1, 2, 3, 4\}$  of Figure 3,

$$\mathcal{B} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}.$$

Note that  $\mathcal{B}$  does not satisfy the path property since the unique 4-path  $E = \{1, 2, 3, 4\}$  is such that  $E \setminus \{4\} = \{1, 2, 3\}$  is not a path.

As we have seen in Example 3, posets antimatroids do not satisfy the path property in general. Thus, a logical question is that if it is required this property to this specific antimatroids if it is possible to obtain some known subclass of the feasible coalition system derived from conjunctive approach of some acyclic permission structure. To establish this relation, we first provide a result on games with permission structures. This result states that the permission structures for which the sets of conjunctive and disjunctive feasible coalitions coincide are exactly the *permission forest* structures.

**Definition 4** An acyclic permission structure  $S$  is a permission forest structure if  $|S^{-1}(i)| \leq 1$ , for all  $i \in N$ .

**Lemma 2** Let  $S$  be an acyclic permission structure. Then  $S$  is a permission forest structure if and only if  $\Phi_S^c = \Phi_S^d$ .

**Proof.** Let  $S$  be an acyclic permission structure. If  $S$  is a permission forest structure then for every  $i \in N$  we have that  $S^{-1}(i) = \emptyset$  or  $S^{-1}(i) \cap E \neq \emptyset$  if and only if  $S^{-1}(i) \subseteq E$ . But then  $\Phi_S^c = \Phi_S^d$ .

If  $S$  is not a permission forest structure then there exists  $j \in N$  with  $|S^{-1}(j)| \geq 2$ . If  $h \in S^{-1}(j)$  then  $\hat{S}^{-1}(j) \setminus \{h\} \in \Phi_S^d \setminus \Phi_S^c$ , implying that  $\Phi_S^c \neq \Phi_S^d$ .  $\square$

The next theorem states that the poset antimatroids satisfying the path property, are exactly those antimatroids that can be obtained as the set of conjunctive or disjunctive feasible coalitions of some permission forest structure. Note that given  $\mathcal{A}$  a poset antimatroid,  $\mathcal{A}$  satisfies the path property if and only if it is verified P1.

**Theorem 3**  $\mathcal{A}$  is a poset antimatroid satisfying the path property if and only if there exists a permission forest structure  $S$  such that  $\mathcal{A} = \Phi_S^c = \Phi_S^d$ .

**Proof.** Suppose that  $\mathcal{A}$  is a poset antimatroid satisfying the path property. Define the permission structure  $S : N \rightarrow 2^N$  by

$$S^{-1}(i) = \{j \in N : \text{there exists a } i\text{-path } E \text{ with } j \in E \text{ and } E \setminus \{i\} \text{ is a } j\text{-path}\}.$$

Since  $\mathcal{A}$  is a poset antimatroid, Lemma 1 implies that there is a unique  $i$ -path for every  $i \in N$ . But then  $S$  is a permission forest structure. We are left to show that  $\Phi_S^c = \mathcal{A}$ . For every  $i \in N$ , let  $H^i$  be the unique  $i$ -path in  $\mathcal{A}$ . Then  $E \in \Phi_S^c$  if and only if  $S^{-1}(i) \subseteq E$ , for all  $i \in E$  if and only if  $H^i \subseteq E$ , for all  $i \in E$  if and only if  $\bigcup_{i \in E} H^i = E$  if and only if  $E \in \mathcal{A}$ . By Lemma 2 we conclude that  $\mathcal{A} = \Phi_S^c = \Phi_S^d$ .

Assume that  $\mathcal{A} = \Phi_S^c = \Phi_S^d$  for some acyclic permission structure  $S$ . By Theorem 1,  $\Phi_S^c$  is a poset antimatroid. By Theorem 2 it is satisfied the path property. So,  $\mathcal{A}$  is a poset antimatroid satisfying the path property.  $\square$

### 3 Restricted games on antimatroids

A cooperative game on a finite set of players  $N$  is a function  $v : 2^N \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ . According to the previous section, the interior of a coalition in an antimatroid is the largest part of  $E$  that is active or feasible. In other words and taking into account that the antimatroid structure limits the possibilities of coalitional formation in a TU-game, the interior of  $E$  are those players of  $E$  that are allowed to cooperate. Therefore, the interior is what Gilles *et al.* (1992) and Gilles and Owen (1999) called (conjunctive or disjunctive) sovereign part in a permission structure. So, we define a restricted game on an antimatroid for a coalition  $E$  as the value obtained on the interior of  $E$ .

**Definition 5** Let  $\mathcal{A}$  be an antimatroid on  $N$ . If  $v$  is a cooperative game then the restricted game on  $\mathcal{A}$  is defined by  $v_{\mathcal{A}}(E) = v(\text{int}(E))$ .

Next, we state some properties of restricted games on antimatroids.

**Proposition 3** Let  $\mathcal{A}$  be an antimatroid on  $N$  and let  $v$  be a monotone game. The following sentences hold

1. The restricted game  $v_{\mathcal{A}}$  is monotone, moreover given  $E, F \subseteq N$  such that  $\text{int}(E) \subseteq \text{int}(F)$  then  $v_{\mathcal{A}}(E) \leq v_{\mathcal{A}}(F)$ .
2. If  $v$  is superadditive then  $v_{\mathcal{A}}$  is superadditive, moreover for  $E, F \subseteq N$  with  $\text{int}(E) \cap \text{int}(F) = \emptyset$  it is satisfied that  $v_{\mathcal{A}}(E \cup F) \geq v_{\mathcal{A}}(E) + v_{\mathcal{A}}(F)$ .
3. If  $v$  is balanced then  $v_{\mathcal{A}}$  is balanced, satisfying that  $C(v) \subseteq C(v_{\mathcal{A}})$ . Moreover, if  $x \in C(v_{\mathcal{A}})$  then  $x(\text{int}(E)) \geq v_{\mathcal{A}}(E)$ , for all  $E \subseteq N$ .
4. If  $v$  is totally balanced then  $v_{\mathcal{A}}$  is totally balanced.

**Proof.** Notice that to prove the two first parts it is only necessary to assume that  $v$  is monotone on  $\mathcal{A}$ , but it is not sufficient in 3 and 4.

1. Let  $E, F \subseteq N$  such that  $\text{int}(E) \subseteq \text{int}(F)$ , then

$$v_{\mathcal{A}}(E) = v(\text{int}(E)) \leq v(\text{int}(F)) = v_{\mathcal{A}}(F).$$

Applying the property I3, we get that the restricted game  $v_{\mathcal{A}}$  is monotone.

2. Let  $E, F \subseteq N$  with  $\text{int}(E) \cap \text{int}(F) = \emptyset$ , then

$$\begin{aligned} v_{\mathcal{A}}(E \cup F) &= v(\text{int}(E \cup F)) \geq v(\text{int}(E) \cup \text{int}(F)) \\ &\geq v(\text{int}(E)) + v(\text{int}(F)), \end{aligned}$$

the first inequality is obtained because  $\text{int}(E) \cup \text{int}(F) \subseteq \text{int}(E \cup F)$  since the set  $\text{int}(E) \cup \text{int}(F) \subseteq E \cup F$  is a feasible coalition in  $\mathcal{A}$ .

3. If  $v$  is a balanced game then  $C(v) \neq \emptyset$ , i.e., there exists an  $x \in C(v)$ . We have to prove that  $x \in C(v_{\mathcal{A}})$ . First,  $x(N) = v(N) = v_{\mathcal{A}}(N)$ , because  $\text{int}(N) = N$ . Next, since  $v$  is monotone,  $x \in C(v)$  we have  $x(E) \geq v(E) \geq v(\text{int}(E)) = v_{\mathcal{A}}(E)$ , for all  $E \subseteq N$ .

4. Let  $E \subseteq N$ ,  $E \neq \emptyset$ . As  $v$  is totally balanced it is satisfied that  $C(v_E) \neq \emptyset$ , there exists  $x \in \mathbb{R}^E$  such that  $x(F) \geq v(F)$ , for all  $F \subseteq E$  and  $x(E) = v(E)$ .

If  $E \in \mathcal{A}$  then  $\mathcal{A}|_E$  is also an antimatroid on  $E$  whose interior operator is the restriction of  $\mathcal{A}$  to  $E$ , therefore  $(v_E)_{\mathcal{A}|_E} = (v_{\mathcal{A}})_E$ . Using sentence 3 we have that  $x \in C((v_E)_{\mathcal{A}|_E})$ , i.e.,  $x \in C((v_{\mathcal{A}})_E)$ .

Consider now that  $E \notin \mathcal{A}$ . In that case  $v_{\text{int}(E)}$  is also balanced, therefore there exists  $x \in C(v_{\text{int}(E)})$  in  $\mathbb{R}^{\text{int}(E)}$  such that  $x(F) \geq v(F)$ , for all  $F \subseteq \text{int}(E)$  and  $x(\text{int}(E)) = v(\text{int}(E))$ . Let  $y \in \mathbb{R}^E$  be the vector given by

$$y_i = \begin{cases} x_i, & \text{if } i \in \text{int}(E) \\ 0, & \text{if } i \in E \setminus \text{int}(E). \end{cases}$$

We have to prove that  $y \in C((v_{\mathcal{A}})_E)$ . First, note that if  $F \subseteq E$  then we obtain  $y(F) \geq v(\text{int}(F)) = v_{\mathcal{A}}(F)$ . Since  $\text{int}(F) \subseteq F \cap \text{int}(E)$ , we have that

$$y(F) = x(F \cap \text{int}(E)) \geq v(F \cap \text{int}(E)) \geq v(\text{int}(F)) = v_{\mathcal{A}}(F).$$

Finally, it remains to show that  $y(E) = v_{\mathcal{A}}(E) = v(\text{int}(E))$ . It is immediate since  $y(E) = x(\text{int}(E)) = v(\text{int}(E))$ .  $\square$

**Proposition 4** *If  $\mathcal{A}$  is an antimatroid on  $N$  with a unique atom (i.e., there exists a unique  $i \in N$  with  $\{i\} \in \mathcal{A}$ ) and  $v$  is monotone then  $v_{\mathcal{A}}$  is monotone, superadditive, balanced and totally balanced. Moreover, if  $\mathcal{A}$  is atomic then  $\mathcal{A} = 2^N$  and  $v = v_{\mathcal{A}}$ .*

**Proof.** Let  $\mathcal{A}$  be an antimatroid on  $N$  with a unique atom  $i_0$ , and let  $v$  be monotone. Monotonicity of  $v_{\mathcal{A}}$  follows from Proposition 3. Let  $E, F \subseteq N$  with  $E \cap F = \emptyset$ . If  $i_0 \notin E \cup F$  then  $\text{int}(E \cup F) = \text{int}(E) = \text{int}(F) = \emptyset$ , and thus  $v(\text{int}(E \cup F)) = v(\text{int}(E)) + v(\text{int}(F)) = 0$ . If  $i_0 \in E \cup F$ , assume without loss of generality that  $i_0 \in E$ . Then  $\text{int}(F) = \emptyset$ , and by monotonicity of  $v$  it holds that  $v(\text{int}(E \cup F)) \geq v(\text{int}(E)) = v(\text{int}(E)) + v(\text{int}(F))$ , showing superadditivity of  $v_{\mathcal{A}}$ .

Let  $x \in \mathbb{R}^N$  be given by  $x_{i_0} = v(N)$  and  $x_i = 0$ , otherwise. Since  $x(N) = v(N)$ ,  $x(E) = v(N) \geq v(\text{int}(E))$  if  $i_0 \in E$ , and  $x(E) = 0 = v(\text{int}(E))$  if  $i_0 \notin E$ , it holds that  $x \in C(v_{\mathcal{A}})$ . So,  $v_{\mathcal{A}}$  is balanced. The atomic case is straightforward.  $\square$

Notice that the fact that the antimatroid has a unique atom means that there is a top player in the permission structure, and if  $\mathcal{A}$  is atomic then all players are in the same level.

Convexity is a particularly desirable property of a game, since it implies balancedness and superadditivity in combination with some geometric properties of the core. However, there exist examples (see Gilles and Owen, 1999) that show that although the game is monotone and convex and  $\mathcal{A}$  is an atomic antimatroid, the restricted game on the antimatroid does not have to be convex.

As it is well-known the set of the *unanimity games*  $\{u_T : T \subseteq N, T \neq \emptyset\}$  is a basis of the vector space  $\Gamma^N$  (set of all TU-games). So, a TU-game  $v$  can be expressed as a linear combination of the unanimity games, i.e.,  $v = \sum_{T \in 2^N \setminus \{\emptyset\}} d_v(T) u_T$  where  $d_v(T)$  is called *dividend* of  $T$  in the game  $v$  and  $d_v(\emptyset) = 0$  (see Harsanyi, 1963).

Our next aim is to obtain the expression of the dividends of the restricted game in terms of the dividends of the original game. For that, we first need some results.

**Proposition 5** *Let  $\mathcal{A}$  be an antimatroid on  $N$  and  $E \subseteq N$ . The restricted game of the unanimity game  $u_E$  on  $\mathcal{A}$  is  $(u_E)_{\mathcal{A}} = \bigvee_{F \in A(E)} u_F$ . Moreover, its dividends are given, for all  $F \subseteq N$ , by*

$$d_{(u_E)_{\mathcal{A}}}(F) = \sum_{\{H \subseteq F : \exists T \in A(E), H \supseteq T\}} (-1)^{|F| - |H|}.$$

*In particular,  $d_{(u_E)_{\mathcal{A}}}(F) = 0$ , if  $F \not\supseteq E$  and  $d_{(u_E)_{\mathcal{A}}}(F) = 1$ , if  $F \in A(E)$ .*

**Proof.** Let  $H \subseteq N$ , then

$$(u_E)_{\mathcal{A}}(H) = u_E(\text{int}(H)) = \begin{cases} 1, & \text{if } E \subseteq \text{int}(H) \\ 0, & \text{otherwise.} \end{cases}$$

If  $E \subseteq \text{int}(H) \in \mathcal{A}$  then there exists  $F \in A(E)$ , such that  $F \subseteq \text{int}(H)$ , therefore  $u_F(H) = 1$ . In this case  $\bigvee_{F \in A(E)} u_F(H) = 1$ . Otherwise,  $H$  does not contain any feasible hull of  $E$  and therefore  $u_F(H) = 0$ , for all  $F \in A(E)$ , which implies that  $\bigvee_{F \in A(E)} u_F(H) = 0$ .

Let us consider  $F \subseteq N$ ,

$$\begin{aligned} d_{(u_E)_A}(F) &= \sum_{H \subseteq F} (-1)^{|F|-|H|} (u_E)_A(H) = \sum_{H \subseteq F} (-1)^{|F|-|H|} \bigvee_{T \in A(E)} u_T(H) \\ &= \sum_{\{H \subseteq F: \exists T \in A(E), H \supseteq T\}} (-1)^{|F|-|H|}. \end{aligned}$$

If  $F \not\supseteq E$  then there is not any  $H \subseteq F$  which contains some feasible hull of  $E$  and hence  $d_{(u_E)_A}(F) = 0$ . Finally, if  $F \in A(E)$ , by definition, the unique term in the above formula is obtained for  $H = F$  and therefore  $d_{(u_E)_A}(F) = 1$ .  $\square$

Next, we consider the particular case from a poset antimatroid. We will take into account that for these antimatroids each coalition has a unique feasible hull (the set obtained as the intersection of all feasible coalitions that contain the coalition) since the intersection is a closed operation.

**Corollary 1** *Let  $\mathcal{A}$  be a poset antimatroid on  $N$  and  $E \subseteq N$ . The restricted game  $(u_E)_A = u_T$ , where  $T$  is the unique feasible hull of  $E$ . Moreover, all dividends are null except  $d_{(u_E)_A}(T) = 1$ .*

**Proof.** By Proposition 5, if  $F = T$  then  $d_{(u_E)_A}(T) = 1$ . Moreover, if  $F \not\supseteq E$  then  $d_{(u_E)_A}(F) = 0$ . Finally, if  $F \supseteq E$  and  $F \neq T$ , since  $F \not\supseteq T$ , we have that

$$d_{(u_E)_A}(F) = \sum_{T \subseteq H \subseteq F} (-1)^{|F|-|H|} = \sum_{h=|T|}^{|F|} \binom{|F|-|T|}{h-|T|} (-1)^{|F|-h} = (1-1)^{|F|-|T|} = 0.$$

This completes the proof.  $\square$

Using the dividends of the restricted games of unanimity games we can obtain the dividends of any restricted game on the antimatroid.

**Theorem 4** *Let  $\mathcal{A}$  be an antimatroid on  $N$  and let  $v$  be a cooperative game. The dividends of the game  $v_A$  are, for all  $F \subseteq N$ ,*

$$d_{v_A}(F) = \begin{cases} \sum_{E \subseteq F} d_{(u_E)_A}(F) d_v(E), & \text{if } F \in \mathcal{A} \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** We first suppose that  $F \notin \mathcal{A}$ , then

$$\begin{aligned} d_{v_A}(F) &= \sum_{E \subseteq F} (-1)^{|F|-|E|} v(int(E)) \\ &= \sum_{\{E \subseteq F: E \in \mathcal{A}\}} \left( \sum_{\{H \subseteq F: int(H)=E\}} (-1)^{|F|-|H|} \right) v(E). \end{aligned}$$

Observe that given  $E \subseteq F$  such that  $E \in \mathcal{A}$  the possible elements in the set  $\{H \subseteq F: int(H) = E\}$  are those coalitions  $H$  contained in  $F$  that contain  $E$  but that do not contain any augmentation player of  $E$  (by condition A2). Let  $k_E = |F - (E \cup au(E))|$  then

$$\begin{aligned} d_{v_A}(F) &= \sum_{\{E \subseteq F: E \in \mathcal{A}\}} \left( \sum_{p=0}^{k_E} \binom{k_E}{p} (-1)^{|F|-p} \right) v(E) \\ &= (-1)^{|F|} \sum_{\{E \subseteq F: E \in \mathcal{A}\}} (1-1)^{k_E} v(E) = 0. \end{aligned}$$

It is satisfied that  $k_E > 1$  since  $F \notin \mathcal{A}$  and moreover the set  $E \cup [au(E) \cap F]$  is feasible because  $E \cup [au(E) \cap F] = \bigcup_{j \in au(E) \cap F} (E \cup \{j\}) \in \mathcal{A}$ , by the condition A3 and the augmentation player definition.

Suppose that  $F \in \mathcal{A}$ , now we use induction on the cardinal of  $F$ . If  $|F| = 1$ , then  $F = \{i\}$  and we have that

$$d_{v_{\mathcal{A}}}(\{i\}) = \sum_{E \subseteq \{i\}} (-1)^{1-|E|} v_{\mathcal{A}}(E) = -v(int(\emptyset)) + v(int(\{i\})) = v(\{i\}),$$

since by I1  $int(\emptyset) = \emptyset$  and as  $F \in \mathcal{A}$  then  $int(\{i\}) = \{i\}$ . On the other hand,

$$\sum_{E \subseteq \{i\}} d_{(u_E)_{\mathcal{A}}}(\{i\}) d_v(E) = d_{(u_{\emptyset})_{\mathcal{A}}}(\{i\}) d_v(\emptyset) + d_{(u_{\{i\}})_{\mathcal{A}}}(\{i\}) d_v(\{i\}) = v(\{i\}),$$

because  $d_v(\emptyset) = 0$ ,  $d_v(\{i\}) = v(\{i\})$  and by Proposition 5,  $d_{(u_{\{i\}})_{\mathcal{A}}}(\{i\}) = 1$  since  $\{i\}$  is the feasible hull of  $\{i\}$ . Suppose that the result is true if  $|F| \leq k$  and we will prove it if  $|F| = k + 1$ . As  $F \in \mathcal{A}$ ,  $int(F) = F$ ,

$$\sum_{E \subseteq F} d_v(E) = v(F) = v_{\mathcal{A}}(F) = \sum_{E \subseteq F} d_{v_{\mathcal{A}}}(E) = \sum_{E \subsetneq F} d_{v_{\mathcal{A}}}(E) + d_{v_{\mathcal{A}}}(F).$$

By using the induction hypothesis and the case  $E \notin \mathcal{A}$  considered above we have that

$$\begin{aligned} d_{v_{\mathcal{A}}}(F) &= \sum_{E \subseteq F} d_v(E) - \sum_{\{E \subsetneq F : E \in \mathcal{A}\}} d_{v_{\mathcal{A}}}(E) \\ &= \sum_{E \subseteq F} d_v(E) - \sum_{\{E \subsetneq F : E \in \mathcal{A}\}} \left( \sum_{H \subseteq E} d_{(u_H)_{\mathcal{A}}}(E) d_v(H) \right) \\ &= \sum_{E \subseteq F} d_v(E) - \sum_{E \subsetneq F} \left( \sum_{E \subseteq H \subseteq F} d_{(u_E)_{\mathcal{A}}}(H) \right) d_v(E), \\ &= d_v(F) + \sum_{E \subsetneq F} \left( 1 - \sum_{E \subseteq H \subseteq F} d_{(u_E)_{\mathcal{A}}}(H) \right) d_v(E). \end{aligned}$$

As  $d_{(u_F)_{\mathcal{A}}}(F) = 1$  by Proposition 5 and  $F \in \mathcal{A}$ , it remains to prove

$$1 - \sum_{E \subseteq H \subseteq F} d_{(u_E)_{\mathcal{A}}}(H) = d_{(u_E)_{\mathcal{A}}}(F), \quad \forall E \subsetneq F,$$

since  $F$  is the unique feasible hull of  $F$ . Applying Proposition 5, we get

$$\begin{aligned} 1 - \sum_{E \subseteq H \subseteq F} d_{(u_E)_{\mathcal{A}}}(H) &= 1 - \sum_{E \subseteq H \subseteq F} \sum_{\{T \subseteq H : \exists R \in \mathcal{A}(E), T \supseteq R\}} (-1)^{|H|-|T|} \\ &= (-1)^{|F|-|F|} - \sum_{\{T \subsetneq F : \exists R \in \mathcal{A}(E), T \supseteq R\}} \sum_{T \subseteq H \subseteq F} (-1)^{|H|-|T|}. \end{aligned}$$

Taking into account that

$$\sum_{T \subseteq H \subseteq F} (-1)^{|H|-|T|} = \sum_{T \subseteq H \subseteq F} (-1)^{|H|-|T|} - (-1)^{|F|-|T|} = -(-1)^{|F|-|T|}$$

since

$$\sum_{T \subseteq H \subseteq F} (-1)^{|H|-|T|} = (1-1)^{|F|-|T|} = 0, \quad (1)$$

we conclude that

$$\begin{aligned} 1 - \sum_{E \subseteq H \subsetneq F} d_{(u_E)_\mathcal{A}}(H) &= (-1)^{|F|-|F|} + \sum_{\{T \subsetneq F: \exists R \in \mathcal{A}(E), T \supseteq R\}} (-1)^{|F|-|T|} \\ &= \sum_{\{T \subseteq F: \exists R \in \mathcal{A}(E), T \supseteq R\}} (-1)^{|F|-|T|} = d_{(u_E)_\mathcal{A}}(F). \end{aligned}$$

Note that in formula (1) we may take  $|F| - |T| \neq 0$ , because otherwise  $F$  would be a feasible hull of  $E$  and  $\sum_{E \subseteq H \subsetneq F} d_{(u_E)_\mathcal{A}}(H) = 0$ .  $\square$

In particular, in Proposition 5 the dividends of a non-feasible coalition for the restricted game of a unanimity game are also zero.

**Corollary 2** *Let  $\mathcal{A}$  be a poset antimatroid on  $N$  and let  $v$  be a cooperative game. The dividends of the game  $v_\mathcal{A}$  are given, for all  $F \subseteq N$ , by*

$$d_{v_\mathcal{A}}(F) = \sum_{\{E \subseteq F: F=A(E)\}} d_v(E),$$

where here  $A(E)$  denote the unique feasible hull of  $E$ .

**Proof.** It is straightforward obtained using Corollary 1 and Theorem 4. Notice that if  $F \notin \mathcal{A}$  then  $d_{v_\mathcal{A}}(F) = 0$ , since in this case  $F$  is not a feasible hull.  $\square$

In the next result we will show that the dividends can be expressed in terms of only the characteristic function evaluated on the feasible coalitions in the antimatroid. For this, we will use the Möbius inversion formula for partially ordered sets (see Stanley, 1986). Notice that an antimatroid  $\mathcal{A}$  with the inclusion relation is, in particular, a lattice taking into account the following operations,  $E \vee F = E \cup F$ ,  $E \wedge F = \text{int}(E \cap F)$ , for all  $E, F \in \mathcal{A}$ . We denote by  $[E, F]_\mathcal{A}$  the interval of coalitions in  $\mathcal{A}$  that contain  $E$  and are contained in  $F$ . Given  $E \in \mathcal{A}$  the set  $E^+$  is the coalition formed by  $E$  and the augmentation players, i.e.,  $E^+ = E \cup \text{au}(E)$ .

**Lemma 3** *Let  $\mathcal{A}$  be an antimatroid on  $N$ , it holds that*

1. *If  $F \subseteq E$ ,  $E, F \in \mathcal{A}$ , the interval  $[F, E]_\mathcal{A}$  is a Boolean algebra if and only if  $E \setminus F = \text{au}(F) \cap E$ .*
2. *If  $E \in \mathcal{A}$  then  $[E, E^+]_\mathcal{A}$  is a Boolean algebra.*

**Proof.**

1. In the lattice  $\mathcal{A}$ , a coalition  $T$  covers  $F$  if  $T = F \cup \{i\}$ , where  $i \in \text{au}(F)$  (by the augmentation property such  $i$  exists). We prove that  $[F, E]_\mathcal{A}$  is a Boolean algebra if and only if  $E$  can be written as  $E = \bigcup_{i \in \text{au}(F) \cap E} (F \cup \{i\})$ . If  $[F, E]_\mathcal{A}$  is a Boolean algebra then  $F \cup \{i\} \in \mathcal{A}$  for all  $i \in E \setminus F$  and therefore  $i \in \text{au}(F) \cap E$ . The other inclusion is straightforward. So, we can conclude that  $[F, E]_\mathcal{A}$  is a Boolean algebra if and only if  $E \setminus F = \text{au}(F) \cap E$ .

2. For all  $i \in E^+ \setminus E$  it holds that  $i \in \text{au}(E)$  and therefore  $E \cup \{i\} \in \mathcal{A}$ . By property (A3) it is concluded that each coalition of  $[E, E^+]_\mathcal{A}$  is feasible.  $\square$

**Lemma 4** *Let  $\mathcal{A}$  be an antimatroid on  $N$  and  $E, F \in \mathcal{A}$ , with  $F \subseteq E$ . Then its Möbius function is given by*

$$\mu(F, E) = \begin{cases} (-1)^{|E|-|F|}, & \text{if } E \setminus F = \text{au}(F) \cap E \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** It is well-known (Stanley, 1986) that the Möbius function of an interval  $[F, E]_\mathcal{A}$  is zero unless  $E$  is the union of those coalitions that cover  $F$  in the interval. By the proof of Lemma 3, this is equivalent to  $E \setminus F = \text{au}(F) \cap E$ . By Lemma 3 the equality is true if and only if  $[F, E]$  is a Boolean algebra. The Möbius function in the Boolean algebra  $[F, E]$  is  $\mu(F, E) = (-1)^{|E|-|F|}$ . So, we conclude the result.  $\square$

**Theorem 5** Let  $\mathcal{A}$  be an antimatroid on  $N$  and let  $v$  be a cooperative game. The dividends of  $v_{\mathcal{A}}$  are, for all  $E \subseteq N$ ,

$$d_{v_{\mathcal{A}}}(E) = \begin{cases} \sum_{\{F \in \mathcal{A}: E \in [F, F^+]\}} (-1)^{|E|-|F|} v(F), & \text{if } E \in \mathcal{A} \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** For every  $E \subseteq N$  it holds that

$$v_{\mathcal{A}} = \sum_{F \subseteq N} d_{v_{\mathcal{A}}}(F) u_F.$$

By Theorem 4, the dividends of any non-feasible coalition are zero. Given a feasible coalition  $E \in \mathcal{A}$  then  $v_{\mathcal{A}}(E) = v(E) = \sum_{\{F \in \mathcal{A}: F \subseteq E\}} d_{v_{\mathcal{A}}}(F)$ . Using the Möbius inversion formula (Stanley, 1986) applied to  $v$  and  $d_{v_{\mathcal{A}}}$  on  $\mathcal{A}$  we get

$$d_{v_{\mathcal{A}}}(E) = \sum_{\{F \in \mathcal{A}: F \subseteq E\}} \mu(F, E) v(F).$$

Then, replacing the Möbius function obtained in Lemma 4 we have that

$$d_{v_{\mathcal{A}}}(E) = \sum_{\{F \in \mathcal{A}: F \subseteq E, E \setminus F = au(F) \cap E\}} (-1)^{|E|-|F|} v(F).$$

The following equality is immediate

$$\{F \in \mathcal{A} : F \subseteq E, E \setminus F = au(F) \cap E\} = \{F \in \mathcal{A} : E \in [F, F^+]_{\mathcal{A}}\},$$

and by Lemma 3, the interval  $[F, F^+]_{\mathcal{A}}$  is a Boolean algebra.  $\square$

#### 4 The restricted Shapley and Banzhaf values on antimatroids

In this section, we consider the Shapley and Banzhaf values of the restricted game on an antimatroid to obtain new values for a game. Taking into account the cooperation possibilities, the values that we define generalize the Shapley and Banzhaf values defined on permission structures and studied in van den Brink and Gilles (1996), van den Brink (1997), Gilles *et al.* (1992), Gilles and Owen (1999). Given the game  $v$  the Shapley (1953) and Banzhaf (1965) values for the player  $i$  in the game  $v$  are denoted by  $Sh_i(v)$  and  $Ba_i(v)$  respectively and in terms of dividends they are given by

$$Sh_i(v) = \sum_{\{E \subseteq N: i \in E\}} \frac{d_v(E)}{|E|}, \quad Ba_i(v) = \sum_{\{E \subseteq N: i \in E\}} \frac{d_v(E)}{2^{|E|-1}}.$$

The Banzhaf power index is introduced for voting situations. A generalization for arbitrary cooperative games as in, e.g., Dubey and Shapley (1979) and Owen (1975) is considered.

**Definition 6** Let  $\mathcal{A}$  be an antimatroid on  $N$  and let  $v$  be a cooperative game. The restricted Shapley value is defined by  $\overline{Sh}(v, \mathcal{A}) = Sh(v_{\mathcal{A}})$ . The restricted Banzhaf value is given by  $\overline{Ba}(v, \mathcal{A}) = Ba(v_{\mathcal{A}})$ .

The linearity of these values and Theorem 4 imply that, for every  $i \in N$ ,

$$\overline{Sh}_i(v, \mathcal{A}) = \sum_{\{E \in \mathcal{A}: i \in E\}} \frac{d_{v_{\mathcal{A}}}(E)}{|E|}, \quad \overline{Ba}_i(v, \mathcal{A}) = \sum_{\{E \in \mathcal{A}: i \in E\}} \frac{d_{v_{\mathcal{A}}}(E)}{2^{|E|-1}}. \quad (2)$$

From Theorem 5 we can get a formula to compute these values using directly the characteristic function of the original game.

**Theorem 6** Let  $\mathcal{A}$  be an antimatroid on  $N$  and let  $v$  be a cooperative game. We consider the following collections, for  $i \in N$ ,

$$\begin{aligned}\mathcal{A}_i &= \{E \in \mathcal{A} : i \in E\}, \\ \mathcal{A}_i^+ &= \left\{E \in \mathcal{A} : i \in \text{ex}(E), (E \setminus \{i\})^+ = E^+\right\}, \\ \mathcal{A}_i^* &= \left\{E \in \mathcal{A} : i \in \text{au}(E), (E \cup \{i\})^+ \neq E^+\right\}.\end{aligned}$$

Then

1. The restricted Shapley value for player  $i$  is given by

$$\begin{aligned}\overline{Sh}_i(v, \mathcal{A}) &= \sum_{E \in \mathcal{A}_i^+} \frac{(e-1)!(e^+-e)!}{e^+!} [v(E) - v(E \setminus \{i\})] \\ &\quad + \sum_{E \in \mathcal{A}_i \setminus \mathcal{A}_i^+} \frac{(e-1)!(e^+-e)!}{e^+!} v(E) \\ &\quad - \sum_{E \in \mathcal{A}_i^*} \frac{e!(e^+-e-1)!}{e^+!} v(E).\end{aligned}$$

2. The restricted Banzhaf value for player  $i$  is given by

$$\begin{aligned}\overline{Ba}_i(v, \mathcal{A}) &= \sum_{E \in \mathcal{A}_i^+} \frac{1}{2^{e^+-1}} [v(E) - v(E \setminus \{i\})] \\ &\quad + \sum_{E \in \mathcal{A}_i \setminus \mathcal{A}_i^+} \frac{1}{2^{e^+-1}} v(E) - \sum_{E \in \mathcal{A}_i^*} \frac{1}{2^{e^+-1}} v(E),\end{aligned}$$

where  $e = |E|$  and  $e^+ = |E^+|$ .

**Proof.** 1. By Theorem 5 and the formula (2)

$$\overline{Sh}_i(v, \mathcal{A}) = \sum_{\{F \in \mathcal{A} : i \in F\}} \frac{1}{|F|} \left[ \sum_{\{E \in \mathcal{A} : F \in [E, E^+]_{\mathcal{A}}\}} (-1)^{|F|-|E|} v(E) \right].$$

Hence,

$$\overline{Sh}_i(v, \mathcal{A}) = \sum_{E \in \mathcal{A}} \left[ \sum_{\{F \in [E, E^+]_{\mathcal{A}} : i \in F\}} \frac{1}{|F|} (-1)^{|F|-|E|} \right] v(E) = \sum_{E \in \mathcal{A}} c_i(E) v(E),$$

where we denote by  $c_i(E)$  the coefficient of  $v(E)$  in the above sum. We are distinguish two cases:  $i \in E$  and  $i \notin E$ .

If  $i \in E$ , as  $[E, E^+]_{\mathcal{A}}$  is a Boolean algebra then

$$\begin{aligned}c_i(E) &= \sum_{k=e}^{e^+} \binom{e^+-e}{k-e} \frac{(-1)^{k-e}}{k} = \sum_{k=0}^{e^+-e} \binom{e^+-e}{k} \frac{(-1)^k}{k+e} \\ &= \sum_{k=0}^{e^+-e} \binom{e^+-e}{k} (-1)^k \int_0^1 x^{k+e-1} dx \\ &= \int_0^1 x^{e-1} (1-x)^{e^+-e} dx = \frac{(e-1)!(e^+-e)!}{e^+!}.\end{aligned}$$



If  $i \notin E$  then  $i \in au(E)$  otherwise  $c_i(E) = 0$ . In that case,  $[E \cup \{i\}, E^+]$  is a Boolean algebra and hence

$$\begin{aligned} c_i(E) &= \sum_{k=e+1}^{e^+} \binom{e^+ - e - 1}{k - e - 1} \frac{(-1)^{k-e}}{k} = \sum_{k=0}^{e^+ - e - 1} \binom{e^+ - e - 1}{k} \frac{(-1)^{k+1}}{k + e + 1} \\ &= - \sum_{k=0}^{e^+ - e - 1} \binom{e^+ - e - 1}{k} (-1)^k \int_0^1 x^{k+e} dx \\ &= - \int_0^1 x^e (1-x)^{e^+ - e - 1} dx = - \frac{e! (e^+ - e - 1)!}{e^+!}. \end{aligned}$$

Replacing the coefficients in the formula it boils down to

$$\overline{Sh}_i(v, \mathcal{A}) = \sum_{E \in \mathcal{A}_i} \frac{(e-1)! (e^+ - e)!}{e^+!} v(E) - \sum_{\{E \in \mathcal{A} : i \in au(E)\}} \frac{e! (e^+ - e - 1)!}{e^+!} v(E). \quad (3)$$

For every  $E \in \mathcal{A}_i$ , if  $i \in ex(E)$  and  $(E \setminus \{i\})^+ = E^+$ , then  $E \in \mathcal{A}_i^+$ . In that case, we have that  $E \setminus \{i\} \in \mathcal{A}$  and  $c_i(E \setminus \{i\}) = -c_i(E)$ , and therefore we can group both coefficients. If  $E \in \mathcal{A}_i \setminus \mathcal{A}_i^+$  its coefficient appears in the first sum but it can not be grouped. Finally, if none of these possibilities happen then  $E \in \mathcal{A}$ ,  $i \in au(E)$  and  $(E \setminus \{i\})^+ \neq E^+$ , i.e.,  $E \in \mathcal{A}_i^*$ . So, these appear in the second sum.

2. By the above part, we know that

$$\overline{Ba}_i(v, \mathcal{A}) = \sum_{E \in \mathcal{A}} c_i(E) v(E),$$

where in this case

$$c_i(E) = \sum_{\{F \in [E, E^+]_{\mathcal{A}} : i \in F\}} \frac{1}{2^{|F|-1}} (-1)^{|F|-|E|},$$

if  $i \in E$  then

$$c_i(E) = \sum_{k=e}^{e^+} \binom{e^+ - e}{k - e} \frac{(-1)^{k-e}}{2^{k-1}} = \sum_{k=0}^{e^+ - e} \binom{e^+ - e}{k} \frac{(-1)^k}{2^{k+e-1}} = \frac{1}{2^{e^+ - 1}},$$

and if  $i \in au(E)$  then

$$c_i(E) = \sum_{k=e+1}^{e^+} \binom{e^+ - e - 1}{k - e - 1} \frac{(-1)^{k-e}}{2^{k-1}} = - \sum_{k=0}^{e^+ - e - 1} \binom{e^+ - e - 1}{k} \frac{(-1)^k}{2^{k+e}} = \frac{-1}{2^{e^+ - 1}}.$$

Doing the same process as in the case of the Shapley value we get the formula for the Banzhaf value.  $\square$

**Remark 1** Notice that if  $\mathcal{A} = 2^N$  then the formulas obtained in Theorem 6 are equal to the Shapley and Banzhaf values. Moreover, the equation (3) is efficient from the computational point of view and coincides with the equation (11) of Shapley (1953).

To apply the computational method developed in this section we analyze two examples.

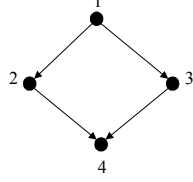


Figure 4

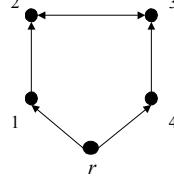


Figure 5

**Example 4** Let  $N = \{1, 2, 3, 4\}$ , given  $v = u_{\{4\}}$  the unanimity game on the coalition  $\{4\}$  and  $\mathcal{S}$  the acyclic permission structure on  $N$  given by (see Figure 4).

$$S(1) = \{2, 3\}, S(2) = S(3) = \{4\}, S(4) = \emptyset.$$

The disjunctive and the conjunctive approaches in the permission structure are the antimatroids considered in Examples 2 and 3 respectively (see Figures 2 and 3).

To compute the restricted Shapley value on  $\mathcal{A}$  and  $\mathcal{B}$  we use the formula (3) since so we only have to calculate  $\mathcal{A}_i$  and  $\{E \in \mathcal{A} : i \in au(E)\}$ . Notice that with this formula, if the game changes but the structure continue being the same the calculations on these sets only are done once. So, the corresponding sets  $\mathcal{A}_i$  belong to the disjunctive approach are given by

$$\begin{aligned} \mathcal{A}_1 &= \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, N\}, \\ \mathcal{A}_2 &= \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, N\}, \\ \mathcal{A}_3 &= \{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, N\}, \\ \mathcal{A}_4 &= \{\{1, 2, 4\}, \{1, 3, 4\}, N\}, \end{aligned}$$

and

$$\begin{aligned} \{E \in \mathcal{A} : 1 \in au(E)\} &= \{\emptyset\}, \\ \{E \in \mathcal{A} : 2 \in au(E)\} &= \{\{1\}, \{1, 3\}, \{1, 3, 4\}\}, \\ \{E \in \mathcal{A} : 3 \in au(E)\} &= \{\{1\}, \{1, 2\}, \{1, 2, 4\}\}, \\ \{E \in \mathcal{A} : 4 \in au(E)\} &= \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}, \end{aligned}$$

thus, applying the above formula we obtain  $\overline{Sh}(v, \mathcal{A}) = (5/12, 1/12, 1/12, 5/12)$  and in a similar way  $\overline{Sh}(v, \mathcal{B}) = (1/4, 1/4, 1/4, 1/4)$ .

Now, the following example shows that the generalization of the model of permission structures to antimatroids has applications in other scopes.

**Example 5** Let  $G = (V, A)$  be a directed graph with root  $r \in V$  and  $c : V \rightarrow \mathbb{R}$  a map on the nodes called capacity function. A directed path with root  $r$  and  $c$ -compatible is a sequence of vertices  $(r, x_1, \dots, x_k)$ , such that  $rx_1 \dots x_k$  is a directed path and such that  $c(r) \geq k, c(x_1) \geq k-1, \dots, c(x_k) \geq 0$ . We interpret  $N = V \setminus \{r\}$  as the clients of a certain source  $r$  and the edges as the directed network that can be established among the clients from the source. The capacity of a vertex is the quantity that can be retransmitted from it. The feasible coalitions that the clients can form are those

in which, without influence of others, can connect to the source through a directed  $c$ -compatible path. The structure that is defined is an antimatroid called capacitated point search,

$$\mathcal{A} = \left\{ E \subseteq N : \forall x \in E, \exists c\text{-compatible path } (r, x_1, \dots, x_k) \right. \\ \left. \text{such that } \{x_1, \dots, x_k\} \subseteq E \text{ and } x_k = x \right\}.$$

These antimatroids do not satisfy the path property that characterize to the antimatroids which derived from a disjunctive approach on an acyclic permission structure, further they are not poset antimatroids, as it is seen in the following example. Consider the directed graph corresponding to Figure 5, whose antimatroid of feasible coalitions is the one described in Example 1. If we suppose that the source offers a discount in the prices of its service to groups that ask for it, depending on their needs, the clients will obtain a profit cooperating among them. We define a game that prizes this profit for each cooperation assuming that it is proportional to the clients number. Without loss of generality we consider  $v(S) = |S| - 1$ , for every non-empty coalition  $S \subseteq N$  and  $v(\emptyset) = 0$ . In a cooperation among several players logically will be taken into account the maximal feasible coalition that the players can form. We used the Shapley value on the game  $v_{\mathcal{A}}$  to determine how to divide the profits of a hypothetical cooperation among the four clients. In order to calculate the payoff of the player 1 through the formula (3) obtained in Theorem 6 we have to calculate before the sets  $\mathcal{A}_i$  and  $\{E \in \mathcal{A} : i \in au(E)\}$ . So, for instance

$$\mathcal{A}_1 = \{\{1\}, \{1, 2\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}, \\ \{E \in \mathcal{A} : 1 \in au(E)\} = \{\emptyset, \{4\}, \{3, 4\}, \{2, 3, 4\}\}.$$

In this way, the division of the profits of the total cooperation,  $v(N) = 3$  is given by  $\overline{Sh}(v, \mathcal{A}) = (5/6, 2/3, 2/3, 5/6)$ .

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